Preface to the Instructor

Goals

This book aims to provide students with the algebraic and trigonometric skills and understanding needed for other coursework and for participating as an educated citizen in a complex society.

Mathematics faculty frequently complain that many students do not read the textbook. When doing homework, a typical student may look only at the relevant section of the textbook or the student solutions manual for an example similar to the homework problem at hand. The student reads enough of that example to imitate the procedure and then does the homework problem. Little understanding may take place.

In contrast, this book is designed to be read by students. The writing style and layout are meant to induce students to read and understand the material. Explanations are more plentiful than typically found in algebra and trigonometry books. Examples of the concepts make the ideas concrete whenever possible.

Exercises and Problems

Each exercise has a unique correct answer, usually a number or a function; most problems have multiple correct answers, usually explanations or examples.

Students learn mathematics by actively working on a wide range of exercises and problems. Ideally, a student who reads and understands the material in a section of this book should be able to do the exercises and problems in that section without further help. However, some of the exercises require application of the ideas in a context that students may not have seen before; many students will need help with these exercises. This help is available from the complete worked-out solutions to all the odd-numbered exercises that appear at the end of each section.

Because the worked-out solutions were written solely by the author of the textbook, students can expect a consistent approach to the material. Furthermore, students will save money by not having to purchase a separate student solutions manual.

This book contains what is usually a separate book called the student solutions manual.

The exercises (but not the problems) occur in pairs, so that an odd-numbered exercise is followed by an even-numbered exercise whose solution uses the same ideas and techniques. A student stumped by an even-numbered exercise should be able to tackle it after reading the worked-out solution to the corresponding odd-numbered exercise. This arrangement allows the text to focus more centrally on explanations of the material and examples of the concepts.

Most students will read the student solutions manual when they are assigned homework, even though they are reluctant to read the main text. The integration of the student solutions manual within this book should encourage students to drift over and also read the main text. To reinforce this tendency, the worked-out solutions to the odd-numbered exercises at the end of each section are intentionally typeset with a slightly less appealing style (smaller type, two-column format, and not right justified) than the main text. The reader-friendly appearance of the main text might nudge students to spend some time there.

Exercises and problems in this book vary greatly in difficulty and purpose. Some exercises and problems are designed to hone algebraic manipulation skills; other exercises and problems are designed to push students to genuine understanding beyond rote algorithmic calculation.

Some exercises and problems intentionally reinforce material from earlier in the book and require multiple steps. For example, Exercise 30 in Section 5.3 asks students to find all numbers x such that

$$\log_5(x+4) + \log_5(x+2) = 2.$$

To solve this exercise, students will need to use the formula for a sum of logarithms as well as the quadratic formula; they will also need to eliminate one of the potential solutions produced by the quadratic formula because it would lead to the evaluation of the logarithm of a negative number. Although such multi-step exercises require more thought than most exercises in the book, they allow students to see crucial concepts more than once, sometimes in unexpected contexts.

The Calculator Issue

The issue of whether and how calculators should be used by students has generated immense controversy.

To aid instructors in presenting the kind of course they want, the symbol appears with exercises and problems that require students to use a calculator.

Some sections of this book have many exercises and problems designed for calculators (for example Section 5.4 on exponential growth), but some sections deal with material not as amenable to calculator use. The text seeks to provide students with both understanding and skills. Thus the book does not aim for an artificially predetermined percentage of exercises and problems in each section requiring calculator use.

Some exercises and problems that require a calculator are intentionally designed to make students realize that by understanding the material, they can overcome the limitations of calculators. As one example among many, Exercise 83 in Section 5.3 asks students to find the number of digits in the decimal expansion of 7^{4000}. Brute force with a calculator will not work with this problem because the number involved has too many digits. However, a few moments' thought should show students that they can solve this problem by using logarithms (and their calculators!).

Regardless of what level of calculator use an instructor expects, students should not turn to a calculator to compute something like $\log 1$*, because then* $\log$ *has become just a button on the calculator.*

The calculator icon can be interpreted for some exercises, depending on the instructor's preference, to mean that the solution should be a decimal approximation rather than the exact answer. For example, Exercise 3 in Section 6.3 asks how much would need to be deposited in a bank account paying 4% interest compounded continuously so that at the end of 10 years the account would contain \$10,000. The exact answer to this exercise is $10000/e^{0.4}$ dollars, but it may be more satisfying to the student (after obtaining the exact answer) to use a calculator to see that approximately \$6,703 needs to be deposited.

For exercises such as the one described in the paragraph above, instructors can decide whether to ask for exact answers or decimal approximations or both (the worked-out solutions for the odd-numbered exercises will usually contain both). If an instructor asks for only an exact answer, then a calculator may not be needed despite the presence of the calculator icon.

Symbolic processing programs such as *Mathematica* and *Maple* offer appealing alternatives to hand-held calculators because of their ability to solve equations and deal with symbols as well as numbers. Furthermore, the larger size, better resolution, and color on a computer screen make graphs produced by such software more informative than graphs on a typical hand-held graphing calculator.

Your students may not use a symbolic processing program because of the complexity or expense of such software. However, easy-to-use free web-based symbolic programs are becoming available. Occasionally this book shows how students can use Wolfram|Alpha, which has almost no learning curve, to go beyond what can be done easily by hand.

Even if you do not tell your students about such free tools, knowledge about such web-based homework aids is likely to spread rapidly among students.

Distinctive Approaches

Half-life and Exponential Growth

Almost all algebra and trigonometry books present radioactive decay as an example of exponential decay. Amazingly, the typical algebra and trigonometry textbook states that if a radioactive isotope has half-life h, then the amount left at time t will equal $e^{-(t \ln 2)/h}$ times the amount present at time 0.

A much clearer formulation would state, as this textbook does, that the amount left at time t will equal $2^{-t/h}$ times the amount present at time 0. The unnecessary use of e and $\ln 2$ in this context may suggest to students that e and natural logarithms have only contrived and artificial uses, which is not the message a textbook should send. Using $2^{-t/h}$ helps students understand the concept of half-life, with a formula connected to the meaning of the concept.

Similarly, many algebra and trigonometry textbooks consider, for example, a colony of bacteria doubling in size every 3 hours, with the textbook then producing the formula $e^{(t\ln 2)/3}$ for the growth factor after t hours. The simpler and more natural formula $2^{t/3}$ seems not to be mentioned in such books. This book presents the more natural approach to such issues of exponential growth and decay.

Algebraic Properties of Logarithms

The base for logarithms in Chapter 5 is arbitrary. Most of the examples and motivation use logarithms base 2 or logarithms base 10. Students will see how the algebraic properties of logarithms follow easily from the properties of exponents.

The crucial concepts of e and natural logarithms are saved for Chapter 6. Thus students can concentrate in Chapter 5 on understanding logarithms (arbitrary base) and their properties without at the same time worrying about grasping concepts related to e. Similarly, when natural logarithms arise naturally in Chapter 6, students should be able to concentrate on issues surrounding e without at the same time learning properties of logarithms.

The initial separation of logarithms and e should help students master both concepts.

Area

Section 2.4 in this book builds the intuitive notion of area starting with squares, and then quickly derives formulas for the area of rectangles, triangles, parallelograms, and trapezoids. A discussion of the effects of stretching either horizontally or vertically easily leads to the familiar formula for the area enclosed by a circle. Similar ideas are then used to find the formula for the area inside an ellipse (without calculus!).

Section 6.1 deals with the question of estimating the area under parts of the curve $y = \frac{1}{x}$ by using rectangles. This easy nontechnical introduction, with its emphasis on ideas without the clutter of the notation of Riemann sums, gives students a taste of an important idea from calculus.

e, The Exponential Function, and the Natural Logarithm

Most algebra and trigonometry textbooks either present no motivation for e or motivate e via continuously compounding interest or through the limit of an indeterminate expression of the form 1^∞; these concepts are difficult for students at this level to understand.

Chapter 6 presents a clean and well-motivated approach to e and the natural logarithm. We do this by looking at the area (intuitively defined) under the curve $y = \frac{1}{x}$, above the x-axis, and between the lines $x = 1$ and $x = c$.

A similar approach to e and the natural logarithm is common in calculus courses. However, this approach is not usually adopted in algebra and trigonometry textbooks. Using basic properties of area, the simple presentation given here shows how these ideas can come through clearly without the technicalities of calculus or Riemann sums.

The approach taken here to the exponential function and the natural logarithm shows that a good understanding of these subjects need not wait until a calculus course.

The approach taken here also has the advantage that it easily leads, as we will see in Chapter 6, to the approximation $\ln(1+h) \approx h$ for $|h|$ small. Furthermore, the same methods show that if r is any number, then

$$(1 + \tfrac{r}{x})^x \approx e^r$$

for large values of x. A final bonus of this approach is that the connection between continuously compounding interest and e becomes a nice corollary of natural considerations concerning area.

Inverse Functions

The unifying concept of inverse functions is introduced in Section 3.4. This crucial idea has its first major use in this book in the definition of $y^{1/m}$ as the number x such that $x^m = y$ (in other words, the function $y \mapsto y^{1/m}$ is the inverse of the function $x \mapsto x^m$; see Section 5.1). The second major use of inverse functions occurs in the definition of $\log_b y$ as the number x such that $b^x = y$ (in other words, the function $y \mapsto \log_b y$ is the inverse of the function $x \mapsto b^x$; see Section 5.2).

Thus students should be comfortable with using inverse functions by the time they reach the inverse trigonometric functions (arccosine, arcsine, and arctangent) in Section 10.1. For students who go on to calculus, this familiarity with inverse functions should help when dealing with inverse operations such as anti-differentiation.

With the approach taken in this book, the statement "$\log_b y = x$ means $b^x = y$" is consistent with the notation used for inverse functions.

This book emphasizes that $f^{-1}(y) = x$ means $f(x) = y$. Thus this book states that to find $f^{-1}(y)$, solve the equation $f(x) = y$ for x.

In contrast, many books at this level unfortunately instruct the reader wanting to find f^{-1} to start with the equation $y = f(x)$, then "interchange the variables x and y to obtain $x = f(y)$", then solve for y in terms of x. This "interchange" method ends up with notation expressing f^{-1} as a function of x.

However, the "interchange" method makes no sense when trying to find the value of an inverse function at a specific number instead of at a variable name. Consider, for example, the problem of finding $f^{-1}(11)$ if f is the function defined by $f(x) = 2x + 3$. The student mechanically following the "interchange" method as it is stated in many books would start with the equation $11 = 2x + 3$ and then interchange x and 11, getting the equation $x = 2 \cdot 11 + 3$. This is, of course, completely wrong.

In contrast, this book does this problem by solving the equation $11 = 2x+3$ for x, getting $x = 4$ and concluding that $f^{-1}(11) = 4$.

The "interchange" method will also be confusing to students when the variables names have meaning. For example, in an applied problem the variables might be t (for time) and d (for distance) rather than x and y, and we might have a function that gives distance in terms of time: $d = f(t)$. The inverse function should then give time in terms of distance: $t = f^{-1}(d)$. Interchanging the variable names here would be quite confusing.

Trigonometry

This book defines $\cos\theta$ and $\sin\theta$ as the first and second coordinates of the radius of the unit circle corresponding to θ (see Section 9.3). In contrast to this definition using only one symbol, many books at this level require students to juggle at least four symbols—θ (or t), x, y, and P—to parse the definitions of the trigonometric functions. These books define $\cos\theta = x$, and students become accustomed to thinking of $\cos\theta$ as the x-coordinate. When students encounter $\cos x$, as often happens within a dozen pages of the initial definition, they think that $\cos x$ is the x-coordinate of ... oops, that is a different use of x. No wonder so many students struggle with trigonometric functions.

This book defines sine and cosine in one section, then defines the tangent function (and the other three trigonometric functions that have less importance) in another section. This gentle approach contrasts with most books that define all six trigonometric functions on the same page. Students have difficulty assimilating so many definitions simultaneously.

This book emphasizes cos, sin, tan *and places little emphasis on* sec, csc, cot.

What to Cover

Different instructors will want to cover different sections of this book. Many instructors will want to cover Chapter 1 (The Real Numbers), even though it should be review, because it deals with familiar topics in a deeper fashion than students may have previously seen.

Some instructors will cover Section 4.3 (Rational Functions) only lightly because graphing rational functions, and in particular finding local minima and maxima, is better done with calculus. Many instructors will prefer to skip Chapter 8 (Sequences, Series, and Limits), leaving that material to a calculus course.

The inverse trigonometric identities (Section 10.2) are given more space in this book than in most books at this level. This material is included not so much for its intrinsic importance but as a way for students to obtain a deeper understanding of the trigonometric functions. Instructors can skip this material or cover it lightly.

Comments Welcome

I seek your help in making this a better book. Please send me your comments and your suggestions for improvements. Thanks!

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